The stability of Couette flow in the presence of an axial magnetic field

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The stability of Couette flow between concentric, co-rotating cylinders in an axial magnetic field is examined for fluids of arbitrary magnetic Prandtl number $Pm = \nu/\eta$, where ν is the kinematic and η the magnetic viscosity of the fluid. It is assumed that the gap spacing d between the cylinders is small compared to the mean radius and that no magnetic disturbances penetrate into the cylinder walls. The critical Taylor number at which non-oscillatory disturbances are marginally stable is determined as a function of the magnetic Prandtl number and the dimensionless parameter $S = (V_a d/\nu)^2$, where V_a is the Alfvén velocity. Asymptotic formulas relating the critical Taylor number to the magnitude of the magnetic field are derived for the limiting conditions of very small and very large magnetic Prandtl number.

1. Introduction

One of the fundamental problems in the field of hydromagnetic stability is that associated with the stability of a viscous, electrically-conducting fluid moving between concentric rotating cylinders in the presence of an axial magnetic field. A special form of this problem was first studied by Chandrasekhar (1953), who treated the case of a weakly conducting fluid. Later Velikhov (1959) considered the case of an inviscid, infinitely-conducting fluid. It is our purpose here to extend these previous investigations by examining the stability of hydromagnetic Couette flow for fluids of arbitrary conductivity. We will show that the magnetic field has a stabilizing effect which depends both on the magnitude of the field and on the magnetic Prandtl number of the fluid.

2. Formulation of the problem

We consider the flow of an incompressible fluid of density ρ, kinematic viscosity ν and electrical conductivity σ between two concentric rotating cylinders of infinite length in the presence of a constant axial magnetic field H_0. Such a flow admits to the stationary hydromagnetic solution

$$V_\theta = Ar + B/r, \quad H_z = H_0, \tag{1}$$

where $A = (\Omega_2 R_2^2 - \Omega_1 R_1^2)/(R_2^2 - R_1^2)$ and $B = -(R_1 R_2)^2 (\Omega_2 - \Omega_1)/(R_2^2 - R_1^2)$, with Ω_1, Ω_2 denoting the angular velocities and R_1, R_2 denoting the radii of the inner and the outer cylinder, respectively. Whenever the gap spacing $d = R_2 - R_1$ is small compared to the mean radius $R_0 = \frac{1}{2}(R_1 + R_2)$ and the cylinders are co-rotating, one can approximate the velocity distribution (1) by its average value

$$\bar{V}_\theta = \Omega_0 R_0 = \tfrac{1}{2}(\Omega_1 + \Omega_2) R_0, \tag{2}$$

and the derivative of V_θ by a constant whose value follows from the linearized form of equation (1) (cf. Chandrasekhar 1953). To examine the stability of this hydromagnetic flow, it is assumed that the stationary conditions (1) are disturbed by the axially-symmetric perturbations

$$\mathbf{v}, \mathbf{h}, p = \{\mathbf{v}(r), \mathbf{h}(r), p(r)\} \exp i(\omega t + kz) \tag{3}$$

in the velocity, the magnetic field and the pressure, respectively. Substituting (1) and (3) into the equations of magnetohydrodynamics and neglecting all quadratic terms in the disturbances, one obtains a set of eight linear differential equations. Upon eliminating the pressure, the axial velocity and axial magnetic field perturbations and assuming that the gap spacing between the cylinders is small compared to R_0 and that the cylinders are co-rotating, these equations reduce to the matrix form

$$\begin{bmatrix} \left(G - i\frac{\omega}{\nu}\right) G & -\frac{2\Omega_0 k^2}{\nu} & \left(\frac{i\mu H_0 k}{\rho\nu}\right) G & 0 \\ -\frac{2A}{\nu} & G - i\frac{\omega}{\nu} & 0 & \frac{i\mu H_0 k}{\rho\nu} \\ \frac{iH_0 k}{\eta} & 0 & G - i\frac{\omega}{\eta} & 0 \\ 0 & \frac{iH_0 k}{\eta} & -\frac{2B}{\eta R_0^2} & G - i\frac{\omega}{\eta} \end{bmatrix} \begin{bmatrix} v_r \\ v_\theta \\ h_r \\ h_\theta \end{bmatrix} = 0, \tag{4}$$

where v_r, v_θ are the components of the velocity perturbation $\mathbf{v}(r)$ and h_r, h_θ the components of the magnetic field perturbation $\mathbf{h}(r)$ in the radial and azimuthal directions, respectively. Here $G = \partial^2/\partial r^2 - k^2$ and $\eta = 1/\mu\sigma$ is the magnetic viscosity (μ denotes the magnetic permeability). Equation (4) can alternatively be written in the tenth-order form

$$G\left[\left(G - i\frac{\omega}{\nu}\right)\left(G - i\frac{\omega}{\eta}\right) + \frac{V_a^2 k^2}{\nu\eta}\right]^2 h_r = \frac{4\Omega_0 A k^2}{\nu^2}\left[\left(G - i\frac{\omega}{\eta}\right)^2 + \left(\frac{V_a k}{\eta}\right)^2\right] h_r, \tag{5}$$

where $V_a = (\mu/\rho)^{\frac{1}{2}} H_0$ is the Alfvén velocity. In deriving equation (5), we have used the relation

$$A = -\left[\frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{(R_1 R_2)^2 (\Omega_2 - \Omega_1)}\right] B \cong -\frac{B}{R_0^2}, \tag{6}$$

which follows from the small-gap approximation, $d \ll R_0$. For vanishing kinematic and magnetic viscosity, equation (5) reduces to the second-order differential equation studied by Velikhov (1959).

We seek a solution of equation (4) corresponding to neutrally stable perturbations (i.e. those disturbances which neither amplify nor decay in time). Furthermore, in order to simplify our analysis, we will consider only those neutrally stable disturbances which are of a non-oscillatory nature. With this restriction, ω can be set equal to zero and equation (4) can, with the aid of (6), be brought to the dimensionless form

$$\begin{bmatrix} L^2 & a^2T & -S\,Pm\,a^2L & 0 \\ -1 & L & 0 & -S\,Pm\,a^2 \\ 1 & 0 & L & 0 \\ 0 & 1 & Pm & L \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0, \tag{7}$$

where $L = \partial^2/\partial x^2 - a^2, \quad x = (r-R_0)/d, \quad a = kd,$

with $y_1 = i\left(\dfrac{2AH_0\,ad^3}{\nu\eta}\right)v_r, \quad y_2 = i\left(\dfrac{H_0\,ad}{\eta}\right)v_\theta, \quad y_3 = \left(\dfrac{2Ad^2}{\nu}\right)h_r, \quad y_4 = h_\theta.$

The Taylor number T, the magnetic Prandtl number Pm and the parameter S measure, respectively, the average rotational speed at the onset of instability, the electrical conductivity of the fluid and the magnitude of the magnetic field. They are defined by

$$T = -\frac{4A\Omega_0 d^4}{\nu^2}, \quad Pm = \frac{\nu}{\eta} = \mu\sigma\nu, \quad S = \left(\frac{V_a d}{\nu}\right)^2 = \frac{\mu}{\rho}\left(\frac{H_0 d}{\nu}\right)^2. \tag{8}$$

An equivalent form of (7), which we will find more useful, is obtained by eliminating the term $-S\,Pm\,a^2L$ in the first row of the matrix by using the third row. This leads to the equation

$$\begin{bmatrix} L^2+S\,Pm\,a^2 & a^2T & 0 & 0 \\ -1 & L & 0 & -S\,Pm\,a^2 \\ 1 & 0 & L & 0 \\ 0 & 1 & Pm & L \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = 0. \tag{9}$$

As $Pm \to 0$ but SPm remains finite, equation (9) reduces to the eighth-order form examined by Chandrasekhar (1953) and by Niblett (1958).

3. Boundary conditions

An appropriate set of ten boundary conditions needed to make equation (9) determinate can be deduced from the elementary requirements that the velocity perturbations vanish at the cylinder walls and that the tangential component of the induced electric field and the normal component of the magnetic intensity are continuous there. In general such a determination will involve the simultaneous solution of the hydromagnetic equations within the fluid and a diffusion equation within the cylinder walls. To avoid the mathematical difficulties inherent in such a solution, we will assume that both the radial and the azimuthal components of the magnetic perturbations vanish at the walls. With this assumption and the no-slip requirement on the velocity perturbations, we obtain the boundary conditions

$$y_1 = \partial y_1/\partial x = y_2 = y_3 = y_4 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{10}$$

4. Solution of the characteristic-value problem

The characteristic-value problem (9) represents a set of four simultaneous differential equations with constant coefficients. Such a set of equations can in principle be solved exactly by expanding the dependent variables in terms of exponential functions and finding the roots of the resultant characteristic polynominal. Unfortunately, in the present problem such a procedure would necessitate finding the ten roots of a tenth-order polynominal, a task requiring a prohibitive amount of computational work. To avoid such difficulties, we will instead employ the Galerkin method (Kantorovich & Krylov 1958). This approximation technique is particularly well suited for problems such as the present one, where it is of more interest to find an approximation to the lowest characteristic value than to obtain detailed knowledge of the characteristic functions. The Galerkin method has been applied previously by Di Prima (1960) to the two simultaneous equations associated with the hydrodynamic stability of flow between concentric cylinders and by Kurzweg (1961) to the three simultaneous equations associated with the hydromagnetic stability of weakly conducting flows.

To apply the Galerkin method to the four simultaneous equations (9) we approximate the dependent variables by the finite series

$$\left.\begin{aligned} \tilde{y}_1 &= \sum_{l=1}^{n} A_l \phi_l, \quad \tilde{y}_2 = \sum_{l=1}^{n} B_l \chi_l, \\ \tilde{y}_3 &= \sum_{l=1}^{n} C_l \psi_l, \quad \tilde{y}_4 = \sum_{l=1}^{n} D_l \zeta_l \end{aligned}\right\} \tag{11}$$

and then demand that the error found by substituting these approximations into each of the equations (9) be orthogonal in the interval $-\frac{1}{2} < x < \frac{1}{2}$ to the function occurring as the highest derivative in the differential equation in question. Such a procedure yields $4n$ algebraic equations for the $4n$ expansion coefficients A_l, B_l, C_l, D_l. The requirement that these coefficients have non-trivial values leads to the secular relation

$$\begin{vmatrix} \langle\phi_l(L^2+S\,Pm\,a^2)\,\phi_m\rangle & a^2T\langle\phi_l\chi_m\rangle & 0 & 0 \\ -\langle\chi_l\phi_m\rangle & \langle\chi_l L\chi_m\rangle & 0 & -S\,Pm\,a^2\langle\chi_l\zeta_m\rangle \\ \langle\psi_l\phi_m\rangle & 0 & \langle\psi_l L\psi_m\rangle & 0 \\ 0 & \langle\zeta_l\chi_m\rangle & Pm\langle\zeta_l\psi_m\rangle & \langle\zeta_l L\zeta_m\rangle \end{vmatrix} = 0, \tag{12}$$

where each term of this determinant represents an $n \times n$ array of elements $(l, m = 1, 2, \ldots, n)$ and

$$\langle f(x)\rangle \equiv \int_{-\frac{1}{2}}^{+\frac{1}{2}} f(x)\,dx.$$

Although the Galerkin method places no restrictions on the form of the trial functions $\phi_l, \chi_l, \psi_l, \zeta_l$ other than that they satisfy the boundary conditions (10), it can be shown from the constant-coefficient form of equation (9) and the symmetry of the boundary conditions (10) that the approximations (11) corresponding to the lowest mode of instability are symmetric about $x = 0$ and vanish nowhere in the interval $-\frac{1}{2} < x < \frac{1}{2}$. It is expected that such symmetric functions can be adequately approximated by a single-term expansion ($n = 1$) of the series (11). Using such an expansion, we obtain the first-order Galerkin approximation

$$T = -\frac{\langle\phi_1(L^2+S\,Pm\,a^2)\,\phi_1\rangle\langle\chi_1 L\chi_1\rangle}{a^2\langle\phi_1\chi_1\rangle^2}\left[\frac{1+S\,Pm\,a^2M(a)}{1+S\,Pm^2a^2N(a)}\right], \tag{13}$$

where

$$M(a) = \frac{\langle\chi_1\zeta_1\rangle^2}{\langle\chi_1 L\chi_1\rangle\langle\zeta_1 L\zeta_1\rangle} \quad\text{and}\quad N(a) = \frac{\langle\chi_1\zeta_1\rangle\langle\phi_1\psi_1\rangle\langle\psi_1\zeta_1\rangle}{\langle\phi_1\chi_1\rangle\langle\psi_1 L\psi_1\rangle\langle\zeta_1 L\zeta_1\rangle}.$$

The value of T found by minimizing (13) with respect to the wave-number at constant S and Pm represents the critical Taylor number T_c. The associated critical wave-number is denoted by a_c.

5. Asymptotic relations

The behaviour of T_c as a function of the magnetic field in the limits of very small and very large magnetic Prandtl number may be deduced from (13) without an explicit evaluation of the definite integrals in the equation. As $Pm \to 0$ but $S\,Pm$ remains finite, the term containing the square of Pm in the denominator of (13) becomes small compared to one and the secular relation reduces to

$$T = -\frac{\langle\phi_1(L^2+S\,Pm\,a^2)\,\phi_1\rangle\langle\chi_1 L\chi_1\rangle}{a^2\langle\phi_1\chi_1\rangle^2}[1+S\,Pm\,a^2M(a)]. \tag{14}$$

This equation represents the small-Pm-theory approximation to the general stability problem and should yield results in agreement with those obtained by Niblett (1958) and Chandrasekhar (1961). It will be noted that the effect of the magnetic field enters the equation only through the parameter $S\,Pm$, which is recognized to be the square of the Hartmann number. Since the Hartmann number represents the ratio of Ohmic to viscous dissipation, it is clear that the only effect of the magnetic field in the limit of small Pm is to hinder the onset of instability by Ohmic dissipation. Chandrasekhar (1961) has shown that the critical Taylor number T_c is directly proportional to $S\,Pm$ and that the critical wave-number a_c approaches zero as $S\,Pm$ becomes large. This linear relationship between T_c and $S\,Pm$ readily follows from (14) by allowing $a_c \to 0$. It has the asymptotic form

$$T_c = -\frac{\langle\phi_1(D^4+\Delta)\,\phi_1\rangle\langle\chi_1 D^2\chi_1\rangle}{\Delta\langle\phi_1\chi_1\rangle^2}\left[1+\frac{\Delta\langle\chi_1\zeta_1\rangle^2}{\langle\chi_1 D^2\chi_1\rangle\langle\zeta_1 D^2\zeta_1\rangle}\right]S\,Pm = c_1 S\,Pm, \tag{15}$$

where $\Delta = (S\,Pm\,a_c^2) = \left[\frac{\langle\phi_1 D^4\phi_1\rangle\langle\chi_1 D^2\chi_1\rangle\langle\zeta_1 D^2\zeta_1\rangle}{\langle\phi_1\phi_1\rangle\langle\chi_1\zeta_1\rangle^2}\right]^{\frac{1}{2}}$ and $D = \frac{\partial}{\partial x}$.

A second limiting form of equation (13) is found by allowing Pm to approach infinity. The terms involving Pm now predominate and (13) reduces to

$$T_c = -\left[\frac{\langle\phi_1\phi_1\rangle\langle\chi_1\zeta_1\rangle\langle\psi_1 D^2\psi_1\rangle}{(\phi_1\chi_1\rangle\langle\phi_1\psi_1\rangle\langle\psi_1\zeta_1\rangle}\right]S = c_2 S. \tag{16}$$

This relation is seen to be independent of the electrical conductivity of the fluid, thus indicating that the stabilization is due to the elastic restoring force of frozen-in magnetic fields. The same linear dependence of T_c on S has been found by Velikhov (1959).

6. Numerical results

Values for the critical Taylor number may be obtained by evaluating equation (13) for a set of trial functions satisfying the symmetry and boundary conditions of the problem. Although there are an infinite number of such functions, we consider here only the elementary polynominals

$$\phi_1 = (1-4x^2)^2, \quad \chi_1 = \psi_1 = \zeta_1 = (1-4x^2). \tag{17}$$

The functions ϕ_1, χ_1 and ζ_1 have been used previously (Kurzweg 1961) to evaluate the small-Pm approximation (14). Results of such an evaluation are recorded in

	a_c		T_c	
$(S\,Pm)^{\frac{1}{2}}$	Equation (14)	Niblett (1958)	Equation (14)	Niblett (1958)
0	3·12	3·135 ± 0·015	$1{\cdot}750\times10^3$	$1{\cdot}708\times10^3$
6	2·55	2·565 ± 0·01	$4{\cdot}583\times10^3$	$4{\cdot}489\times10^3$
10	1·685	1·69 ± 0·01	$1{\cdot}103\times10^4$	$1{\cdot}082\times10^4$
20	0·776	0·775 ± 0·01	$4{\cdot}359\times10^4$	$4{\cdot}272\times10^4$
40	0·380	0·39 ± 0·01	$1{\cdot}746\times10^5$	$1{\cdot}70\times10^5$
80	0·188	0·19 ± 0·01	$6{\cdot}987\times10^5$	$6{\cdot}85\times10^5$

TABLE 1. Small-Pm-theory results for the critical wave and Taylor number as a function of the Hartmann number.

table 1. The critical values of T and a are seen to be in good agreement with these obtained by Niblett (1958) via the considerably more involved variational solution of an equivalent eighth-order equation. Using the functions (17) to solve the asymptotic relations (15) and (16), we find $c_1 = 109{\cdot}2$ in comparison to the value 107·2 given by Chandrasekhar (1961) and $c_2 = 10{\cdot}37$ in comparison to the value $\pi^2 \simeq 9{\cdot}87$ given by Velikhov (1959).

Equation (13) has been evaluated for $S = 1$, 10^2, and 10^4 using the trial functions (17). The results are summarized in figures 1 and 2, together with the values predicted by the small Pm theory and the non-dissipative theory. The three values of the magnetic field were chosen so as to fall into the regions defined by $1750 >, =, < 10{\cdot}37S$. $S = 1$ represents a weak field where the flow is more stable at low magnetic Prandtl number than at large Pm. $S = 10^2$ represents an intermediate field where the critical Taylor number for a non-conducting fluid is approximately equal to that for an infinitely conducting fluid and $S = 10^4$ represents a large field where the flow is more stable at infinite Pm than at zero magnetic Prandtl number.

As shown in figure 1, the flow is most stable for values of Pm near one. This stability maximum is quite pronounced for an intermediate magnetic field but tends to vanish for both very large and very small fields. The maximum stability point is shifted to lower values of Pm as S is increased, ranging from $Pm = 0{\cdot}8$ at $S = 1$ to $Pm = 0{\cdot}4$ at $S = 10^4$. The stability maximum as S becomes infinite occurs near $Pm = 0{\cdot}1$. This is the point where the asymptotic relations (15) and (16) yield equal values for T_c.

The critical wave-number of the disturbances (figure 2) decreases for increasing magnetic field and magnetic Prandtl number, ranging from $a_c = 3{\cdot}12$ at $Pm = 0$ to $a_c = 0$ as Pm approaches infinity. An interesting feature of the (a_c, Pm)-curves

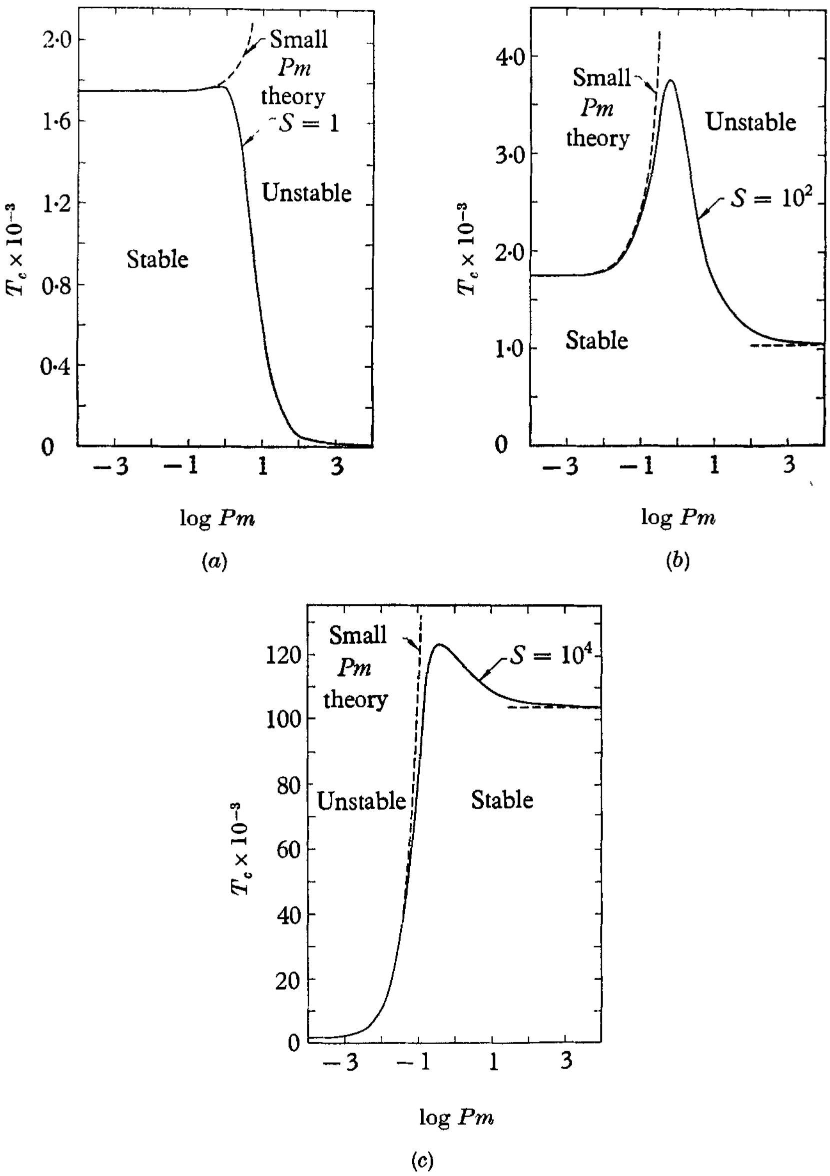

FIGURE 1. The variation of the critical Taylor number as a function of the magnetic Prandtl number for several values of the magnetic field. (*a*) $S = 1$; (*b*) $S = 10^2$; (*c*) $S = 10^4$.

is the kink occurring near $Pm = 0{\cdot}1$ for a strong magnetic field ($S = 10^4$). This kink represents the transition from a dissipative to a non-dissipative instability mode.

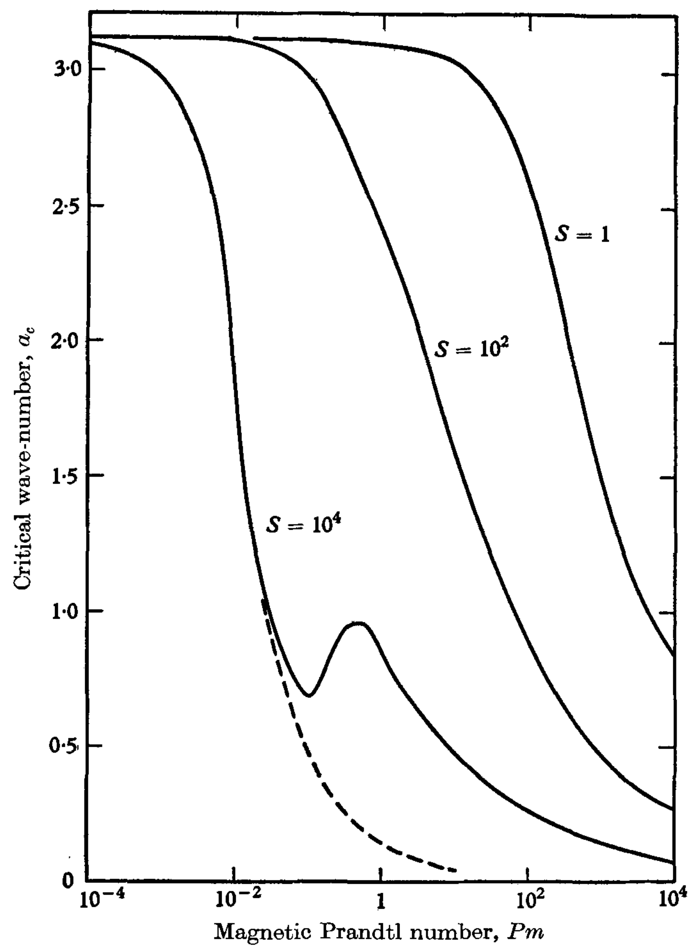

FIGURE 2. The variation of the critical wave-number as a function of the magnetic Prandtl number. The dashed curve represents the small Pm theory results for $S = 10^4$.

7. Concluding remarks

We have examined the hydromagnetic stability of Couette flow between-co-rotating cylinders for fluids of arbitrary magnetic Prandtl number and have determined the approximate values of the critical Taylor number for the case where the instability is characterized by stationary secondary flow. It is found that an increase in the magnetic field always produces an increase in stability of the flow, despite the fact that the mechanisms responsible for this stablization are quite different for fluids of small and large electrical conductivity. For sufficiently large magnetic fields, the results indicate that the Taylor number is directly proportional to the square of the Hartmann number for $Pm < 0{\cdot}1$ and directly proportional to the parameter S for $Pm > 0{\cdot}1$. Several interesting extensions of the present investigation readily suggest themselves. These include the possibility of an exact solution of the characteristic value problem, a consideration of overstable modes of instablity (Re $\omega \neq 0$) and an examination of the counter-rotating cylinder case.

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